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Dual characterizations of the set containments with strict cone-convex inequalities in Banach spaces

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Abstract We give characterizations of the containment of a convex set either in an arbitrary convex set or in a set described by reverse cone-convex inequalities in Banach spaces. The convex sets under consideration are the solution sets of an arbitrary number of cone-convex inequalities, which can be either weak or strict inequalities. These characterizations provide ways of verifying the containments either by comparing their corresponding dual cones or by checking the consistency of suitable associated systems. Particular cases of dual characterizations of set containments have played key roles in solving large scale knowledge-based data classification problems, where they are used to describe the containments as inequality constraints in optimization problems. The concept of evenly convex set is used to derive the dual conditions, characterizing the set containments.

Keywords Set containment · Convex function · Dual cone · Semi-infinite system · Conjugation · Existence theorem

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1 Introduction

Let X and Z be Banach spaces and $\mathcal{S}$ be a closed convex cone in Z. Consider the sets

$$F := \{x \in X : f_i(x) \in -int\mathcal{S},\ \forall\ i \in \mathcal{I};\ \ f_j(x) \in -\mathcal{S},\ \forall\ j \in \mathcal{J}\},$$

and

$$G := \{x \in X : g_t(x) \in -\mathcal{S},\ \forall\ t \in W_1;\ \ h_t(x) \in \mathcal{S},\ \forall\ t \in W_2\},$$

where $\mathcal{I},\ \mathcal{J},\ W_1,\ W_2$ are index sets, $\mathcal{I}\cap\mathcal{J} = \emptyset,\ \mathcal{I}\cup\mathcal{J} \neq \emptyset,\ W_1 \cap W_2 = \emptyset,\ W_1 \cup W_2 \neq \emptyset$, and all functions are $\mathcal{S}$-convex from X to Z. The set containment problem that is studied in

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this paper, consists of deciding whether $F \subset G$ or not. Dual characterizations of such set containments have played a key role in solving large scale knowledge-based data classification problems, where they are used to describe the containments as inequality constraints in optimization problems (see, e.g., [4,7,10,11]).

Various extensions of the set containment problem to more general situations have been obtained in [7] and [11], by means of mathematical programming theory and conjugacy theory, respectively, where $\mathcal{S} \neq \emptyset$ (i.e. without strict inequalities). More recently, dual characterizations by allowing the systems defining F and G to contain strict inequalities in $\mathbb{R}^n$ were established in [5]. In this paper, we establish dual characterizations by allowing the systems defining F and G to contain strict inequalities in Banach spaces. In fact, we generalize the results were obtained in [5] to Banach spaces.

The main basic tool in our approach in deriving the dual characterizations is the association of two dual cones in $X^* \times \mathbb{R}$, say K and M, such that $F \subset G$ if and only if $M \subset K$. Since $M \subset K$ can be interpreted as a dual condition, the verification of the set containment reduces to the effective calculus of the corresponding dual cones. In the case where F is the intersection of a family of open convex sets: $\{x \in X : f_j(x) \in -\mathcal{S}\}$ $(j \in \mathcal{J})$ with a family of closed convex sets: $\{x \in X : f_j(x) \leq 0\}$ $(j \in \mathcal{J})$, F turns out to be an evenly convex set [3], represented by means of convex inequality systems.

The layout of the paper is as follows. In Sect. 2, we collect definitions, notations and preliminary results that will be used later in this paper. In Sect. 3, we obtain some feasibly rules for some closed convex sets and calculus rules for their dual cones. In Sect. 4, we develop calculus rules which are similar to those obtained in Sect. 3 for dual cones of evenly convex sets. In Sect. 5, we give some properties of positively homogenous convex functions and state general existence theorems for convex systems which contain strict reverse-convex inequalities.

2 Preliminaries

We start this section by fixing the notations and preliminaries that will be used later in this paper. Let X and Z be Banach spaces and $\mathcal{S}$ be a closed convex cone in Z. The continuous dual space of X will be denoted by X^*. For a set $\mathcal{A} \subset X^*$, the weak*-closure (resp. closure) of $\mathcal{A}$ will be denoted by $w^* - cl\mathcal{A}$ (resp. $cl\mathcal{A}$), and the weak*-interior of $\mathcal{A}$ will be denoted by $w^* - int\mathcal{A}$.

Given a set $\mathcal{A} \subset X$, we shall denote by $int\mathcal{A}$, $bd\mathcal{A}$, $co\mathcal{A}$ and $coneco\mathcal{A}$ the interior, the boundary, the convex hull and the convex cone generated by $\mathcal{A}$, respectively.

Fenchel [3] defined the class of *evenly convex sets* as the intersection of open half spaces. The evenly convex hull of $\mathcal{A}$, $eco\mathcal{A}$ is the smallest evenly convex set which contains $\mathcal{A}$ (i.e. it is the intersection of all open half spaces which contain $\mathcal{A}$).

The *support function* of $\mathcal{A}$ is defined by

$$\sigma_{\mathcal{A}}(x^*) = \sup_{x \in \mathcal{A}} x^*(x) \quad (x^* \in X^*),$$

and the *indicator function* of $\mathcal{A}$ is defined by

$$\delta_{\mathcal{A}}(x) = \begin{cases} 0, & x \in \mathcal{A} \\ +\infty, & x \notin \mathcal{A}. \end{cases}$$

The *epigraph* of a function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$, $epif$, is defined by

$$epif = \{(x, r) \in X \times \mathbb{R} : x \in domf,\ f(x) \leq r\},$$

where the domain of f, $domf$, is given by

$$domf = \{x \in X : f(x) < +\infty\}.$$

Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semi continuous (lsc) convex function. Then the *conjugate function* of f, denoted by f^*, such that $f^* : X^* \longrightarrow \mathbb{R} \cup \{+\infty\}$, is given by

$$f^*(x^*) = sup\{x^*(x) - f(x) : x \in domf\}, \quad (x^* \in X^*).$$

The *subdifferential* of f at a, $\partial f(a)$, is given by

$$\partial f(a) = \{x^* \in X^* : x^*(x - a) \leq f(x) - f(a), \ \forall\, x \in X\}.$$

For a closed convex cone K in X, define K^+ by

$$K^+ := \{\lambda \in X^* : \lambda(x) \geq 0, \ \forall\, x \in K\}.$$

Let $\{Y_\alpha : \alpha \in \mathcal{B}\}$ be a family of topological spaces, where $\mathcal{B}$ is an index set. consider their products

$$Y := \prod_{\alpha \in \mathcal{B}} Y_\alpha = \{x : \mathcal{B} \longrightarrow \cup_{\alpha \in \mathcal{B}} Y_\alpha : \ x := (x(\alpha))_{\alpha \in \mathcal{B}}, \ x(\alpha) \in Y_\alpha, \ \forall\, \alpha \in \mathcal{B}\}.$$

Denote $x(\alpha) = x_\alpha$, for all $\alpha \in \mathcal{B}$, and $x = (x_\alpha)_{\alpha \in \mathcal{B}}$. Consider the projections $P_\alpha : Y \to Y_\alpha$ defined by $P_\alpha x = x_\alpha$, for all $x \in Y$.

The space Y endowed with the weakest topology which makes each projection continuous is called the product topological space of the topological spaces Y_α, $\alpha \in \mathcal{B}$. Thus, a basis for the product topology is given by the sets of the form $\prod_{\alpha \in \mathcal{B}} D_\alpha$, where D_α is an open subset in Y_α, for all $\alpha \in F$, and $D_\alpha = Y_\alpha$, for all $\alpha \in Y \backslash F$, where F is a finite subset of $\mathcal{B}$. Also, a fundamental neighborhood system of an element $x = (x_\alpha)_{\alpha \in \mathcal{B}}$ is given by the sets having the form

$$\mathcal{V}_{F, \{\mathcal{V}_\alpha : \alpha \in F\}}(x) = \{u = (u_\alpha)_{\alpha \in \mathcal{B}} \in Y; \ u_\alpha \in \mathcal{V}_\alpha(x_\alpha), \ \forall\, \alpha \in F\}, \tag{2.1}$$

where, for each $\alpha \in \mathcal{B}$, $\mathcal{V}_\alpha(x_\alpha)$ runs through a fundamental neighborhood system of $x_\alpha \in Y_\alpha$ (see [2,12]).

The following existence theorem for linear inequality systems containing strict inequalities will be generalized later.

Proposition 2.1 ([7], Theorem 3.1). *Let $\mathcal{I}$ and $\mathcal{J}$ be non-empty index sets. The system* $\{a_t'x < b_t, \ t \in \mathcal{I}; \quad a_t'x \leq b_t, \ t \in \mathcal{J}\}$ $(x, \ a_t \in \mathbb{R}^n; \ b_t \in \mathbb{R})$ *is consistent if and only if*

$$0_{n+1} \notin eco[\{(a_t, b_t) : t \in \mathcal{I}\} + \mathbb{R}_+\{(a_t, b_t) : t \in \mathcal{J}\}; \ (0_n, 1)].$$

3 Containments of closed convex sets

We start this section with the definition of the weak dual cone of a closed convex set. Let $F \subset X$ be a non-empty closed convex set. We define the weak dual cone of F as

$$K^{\leq} = \{(\gamma, b) \in X^* \times \mathbb{R}, \quad \gamma(x) \leq b, \ \forall\, x \in F\} = epi\sigma_F.$$

For explanation of some properties of the weak dual cone, we need to obtain conditions of dual characterizations of the existence theorem for the set $F = \{x : f_t(x) \in -\mathcal{S}, \ \forall\, t \in \mathcal{J}\}$.

Proposition 3.1 *Let $F = \{x : f_t(x) \in -\mathcal{S}, \ \forall\, t \in \mathcal{J}\}$, where $f_t : X \longrightarrow Z$ is a continuous and $\mathcal{S}$-convex function $(t \in \mathcal{J})$. Then the following assertions are true.*

(i) $F \neq \emptyset \Leftrightarrow (0,-1) \notin w^* - clco(\bigcup_{t\in\mathcal{J}} \bigcup_{\lambda\in\mathcal{S}^+} epi(\lambda f_t)^*)$.
(ii) $F \neq \emptyset \Rightarrow\ epi\sigma_F = w^* - clco(\bigcup_{t\in\mathcal{J}} \bigcup_{\lambda\in\mathcal{S}^+} epi(\lambda f_t)^*)$.

Proof (i). Let $t \in \mathcal{J},\ \lambda \in \mathcal{S}^+$ and $x^* \in X^*$ be arbitrary. Since $-\lambda f_t(x) \geq 0$ for all $x \in F$, we get

$$(\lambda f_t)^*(x^*) = \sup_{x\in X}[x^*(x) - \lambda f_t(x)] \geq \sup_{x\in F}[x^*(x) - \lambda f_t(x)] \geq \sup_{x\in F} x^*(x) = \sigma_F(x^*).$$

This inequality with the fact that $epi\sigma_F$ is w^*-closed, gives us $epi\sigma_F \supset w^* - clco(\bigcup_{t\in\mathcal{J}} \bigcup_{\lambda\in\mathcal{S}^+} epi(\lambda f_t)^*)$.

Now, if $F \neq \emptyset$, then clearly $(0,-1) \notin epi\sigma_F$, and so from the above inclusion, we obtain $(0,-1) \notin w^* - clco(\bigcup_{t\in\mathcal{J}} \bigcup_{\lambda\in\mathcal{S}^+} epi(\lambda f_t)^*)$.

Conversely, if $(0,-1) \notin w^* - clco(\bigcup_{t\in\mathcal{J}} \bigcup_{\lambda\in\mathcal{S}^+} epi(\lambda f_t)^*)$, then by the separation theorem there is $(x,\alpha) \in X \times \mathbb{R}$, $(x,\alpha) \neq (0,0)$ such that $-\alpha < 0$,

$$v_t(x) + \gamma_t\alpha \geq 0 \quad \forall\ (v_t,\gamma_t) \in \bigcup_{\lambda\in\mathcal{S}^+} epi(\lambda f_t)^*, \quad \forall\ t \in \mathcal{J}.$$

Let $\overline{x} = \frac{x}{\alpha}$, then it follows that

$$v_t(-\overline{x}) - \gamma_t \leq 0 \quad \forall\ (v_t,\gamma_t) \in \bigcup_{\lambda\in\mathcal{S}^+} epi(\lambda f_t)^*, \quad \forall\ t \in \mathcal{J}.$$

Now, for each $v_t \in dom(\lambda f_t)^*$, we have $v_t(-\overline{x}) - (\lambda f_t)^*(v_t) \leq 0$. Since λf_t is continuous, thus

$$\lambda f_t(-\overline{x}) = (\lambda f_t)^{**}(-\overline{x}) = \sup_{v_t\in \mathrm{dom}\,(\lambda f_t)^*} [v_t(-\overline{x}) - (\lambda f_t)^*(v_t)] \leq 0, \quad \forall\ t \in \mathcal{J}.$$

This implies that $f_t(-\overline{x}) \in -\mathcal{S}$ for each $t \in \mathcal{J}$, and hence $-\overline{x} \in F$, which shows that $F \neq \emptyset$.

(ii). We have already established in part (i) that $epi\sigma_F \supset w^* - clco(\bigcup_{t\in\mathcal{J}} \bigcup_{\lambda\in\mathcal{S}^+} epi(\lambda f_t)^*)$.

To show the converse inclusion, let $(u,\alpha) \notin w^* - clco(\bigcup_{t\in\mathcal{J}} \bigcup_{\lambda\in\mathcal{S}^+} epi(\lambda f_t)^*)$. Since $F \neq \emptyset$, we have $(0,-1) \notin w^* - clco(\cup_{t\in\mathcal{J}} \cup_{\lambda\in\mathcal{S}^+} epi(\lambda f_t)^*)$. Then

$$\mathcal{B}_0 \bigcap [w^* - clco(\bigcup_{t\in\mathcal{J}} \bigcup_{\lambda\in\mathcal{S}^+} epi(\lambda f_t)^*)] = \emptyset,$$

where

$$\mathcal{B}_0 = \{\delta(u,\alpha) + (1-\delta)(0,-1) \in X^* \colon \delta \in [0,1]\}.$$

It is clear that $\mathcal{B}_0$ is a w^*-compact convex set in X^*. Now, by the separation theorem there exists $(x,\beta) \neq (0,0)$ such that for all $\delta \in [0,1]$:

$$[\delta(u,\alpha) + (1-\delta)(0,-1)](x,\beta) < 0,$$

$$v_t(x) + \gamma_t\beta \geq 0, \quad \forall\ (v_t,\gamma_t) \in w^* - clco(\bigcup_{t\in\mathcal{J}} \bigcup_{\lambda\in\mathcal{S}^+} epi(\lambda f_t)^*), \forall\ t \in \mathcal{J}.$$

By letting $\delta = 0$, we get that $\beta > 0$, and by letting $\delta = 1$, we obtain $u(x) + \alpha\beta < 0$. Thus, $u(-\frac{x}{\beta}) > \alpha$. On the other hand, for each $\lambda \in \mathcal{S}^+$ and $t \in \mathcal{J}$, one has

$$v_t(\frac{-x}{\beta}) - \gamma_t \geq 0 \quad \forall\ (v_t,\gamma_t) \in epi(\lambda f_t)^*.$$

This gives us

$$v_t(\frac{-x}{\beta}) - (\lambda f_t)^*(v_t) \leq 0 \quad \forall\, v_t \in dom(\lambda f_t)^*, \quad \forall\, t \in \mathcal{J}.$$

Hence, for each $\lambda \in \mathcal{S}^+$ and $t \in \mathcal{J}$, we have

$$\lambda f_t(-\frac{x}{\beta}) = (\lambda f_t)^{**}(-\frac{x}{\beta}) = \sup_{v_t \in \mathrm{dom}\,(\lambda f_t)^*} [v_t(-\frac{x}{\beta}) - (\lambda f_t)^*(v_t)] \leq 0, \quad \forall\, t \in \mathcal{J}.$$

This implies that $-\frac{x}{\beta} \in F$. This, together with $u(-\frac{x}{\beta}) > \alpha$, gives $(u, \alpha) \notin epi\sigma_F$, which completes the proof. □

Theorem 3.1 *Let Y be a Banach space and $\mathcal{I}$ be an arbitrary index set. Let $K \subset Y$ and $\mathcal{S} \subset Z$ be closed convex cones. Assume that $h_j : X \to Y$, $j = 1, 2, \ldots, m$ $(m \in \mathbb{N})$ is a continuous K-convex function and $f_i : X \to Z$ $(i \in \mathcal{I})$ is a continuous $\mathcal{S}$-convex function. If $F = \{x \in X : f_i(x) \in -\mathcal{S},\ \forall\, i \in \mathcal{I}\} \neq \emptyset$, then the following assertions are equivalent.*

(i) $F \subset \{x \in X : h_j(x) \in K, \quad \forall\, j = 1, 2, \ldots, m\}$.
(ii) *We have* $0 \in epi(\theta h_j)^* + w^* - clco(\bigcup_{i \in \mathcal{I}} \bigcup_{\lambda \in \mathcal{S}^+} epi(\lambda f_i)^*)$, $\forall\, \theta \in K^+$ *and* $\forall\, j = 1, 2, \ldots, m$.

Proof (i) ⇒ (ii). Let $\theta \in K^+$ and $H_\theta := \{x \in X : (\theta h_j)(x) \geq 0, \quad \forall\, j = 1, 2, \ldots, m\}$. It is clear that F is a closed convex set, so (i) implies that $F \subset H_\theta$. Now, $F \subset H_\theta$ if and only if $\theta h_j + \delta_F \geq 0$ for all $j = 1, 2, \ldots, m$. It follows from the definition of $epi(\theta h_j + \delta_F)^*$ and the inequality $\theta h_j + \delta_F \geq 0$ that $0 \in epi(\theta h_j + \delta_F)^*$ $(j = 1, 2, \ldots, m)$.

Since θh_j is a real valued continuous convex function (note that h_j is a continuous K-convex function) and δ_F is a proper lower semi-continuous convex function, then $epi(\theta h_j + \delta_F)^* = epi(\theta h_j)^* + epi\delta_F^*$ (see [8], Lemma 6.7). Since $\delta_F^* = \sigma_F$ and $F \neq \emptyset$, it follows from Proposition 3.1 that

$$epi\sigma_F = w^* - clco(\bigcup_{t \in \mathcal{J}} \bigcup_{\lambda \in \mathcal{S}^+} epi(\lambda f_t)^*).$$

Thus, $0 \in epi(\theta h_j)^* + w^* - clco(\bigcup_{t \in \mathcal{J}} \bigcup_{\lambda \in \mathcal{S}^+} epi(\lambda f_t)^*)$, and hence we conclude that (ii) holds.

(ii) ⇒ (i). For each $\theta \in K^+$ and for each $j = 1, 2, \ldots, m$, it follows from (ii) that there exists $(u_{\theta,j}, \alpha_{\theta,j}) \in epi\sigma_F$ $(j = 1, 2, \ldots, m)$ such that $-(u_{\theta,j}, \alpha_{\theta,j}) \in epi(\theta h_j)^*$ $(j = 1, 2, \ldots, m)$. This implies that if $x \in F$ and $j \in \{1, 2, \ldots, m\}$, then we have $u_{\theta,j}(x) \leq \sigma_F(u_{\theta,j}) \leq \alpha_{\theta,j}$ and $-\alpha_{\theta,j} \geq -u_{\theta,j}(x) - (\theta h_j)(x)$. So, $(\theta h_j)(x) \geq 0$. Since K is a closed convex cone, we get $h_j(x) \in K$ for all $j = 1, 2, \ldots, m$, and hence we have (i). □

The proof of the following corollary is similar to the one of in ([8], Corollary 2.1), and therefore we omit it.

Corollary 3.1 *Let $\mathcal{I}$ be an arbitrary index set and $\mathcal{S} \subset Z$ be a closed convex cone. Let $\alpha_j \in \mathbb{R}$ and $u_j : X \longrightarrow \mathbb{R}$ $(j = 1, 2 \ldots, m)$ be a continuous linear mapping and $f_i : X \longrightarrow Z$ be a continuous $\mathcal{S}$-convex function $(i \in \mathcal{I})$. Assume that the set $\{x \in X : f_i(x) \in -\mathcal{S},\ \forall\, i \in \mathcal{I}\}$ is consistent. Then the following assertions are equivalent.*

(i) $\{x \in X : f_i(x) \in -\mathcal{S},\ \forall\, i \in \mathcal{I}\} \subset \{x \in X : u_j(x) \leq \alpha_j, \quad \forall\, j = 1, 2, \ldots, m\}$.
(ii) $(u_j, \alpha_j) \in w^* - clco(\bigcup_{i \in \mathcal{I}} \bigcup_{\lambda \in \mathcal{S}^+} epi(\lambda f_i)^*), \quad \forall\, j = 1, 2, \ldots, m$.

The following result is a generalization of the non-homogeneous Farkas Lemma in Banach spaces.

Lemma 3.1 *Let $\mathcal{I}$ be an arbitrary index set and $\gamma_i : X \longrightarrow \mathbb{R}$ $(i \in \mathcal{I})$ and $u_j : X \longrightarrow \mathbb{R}$ $(j = 1, 2, \ldots, m)$ be continuous linear maps. Suppose that $a_i,\ \alpha_j \in \mathbb{R}$ $(i \in \mathcal{I},\ j = 1, 2, \ldots, m)$ and the set $\{x \in X : \gamma_i(x) \le a_i,\ \forall\ i \in \mathcal{I}\}$ is consistent. Then the following assertions are equivalent.*

(i) $\{x \in X : \gamma_i(x) \le a_i,\ \forall\ i \in \mathcal{I}\} \subset \{x \in X : u_j(x) \le \alpha_j,\quad \forall\ j = 1, 2, \ldots, m\}$.
(ii) $(u_j, \alpha_j) \in w^* - clconeco\{(\gamma_i, a_i) : i \in \mathcal{I};\ (0, 1)\},\quad \forall\ j = 1, 2, \ldots, m$.

Proof Let $f_i : X \longrightarrow \mathbb{R}$ be defined by $f_i(x) = \gamma_i(x) - a_i$ for each $i \in \mathcal{I}$, and let $\mathcal{S} = \mathbb{R}_+$. Therefore, the condition (i) is equivalent to the following inclusion:

$$\{x : f_i(x) \in -\mathcal{S},\ \forall\ i \in \mathcal{I}\} \subset \{x \in X : u_j(x) \le \alpha_j,\quad \forall\ j = 1, 2, \ldots, m\}.$$

In view of Corollary 3.1, it follows that (i) holds if and only if

$$(u_j, \alpha_j) \in w^* - clco(\bigcup_{i\in\mathcal{I}} \bigcup_{\lambda\in\mathcal{S}^+} epi(\lambda f_i)^*),\quad \forall\ j = 1, 2, \ldots, m.$$

Now, let $\mathcal{A} := w^* - clco(\bigcup_{i\in\mathcal{I}} \bigcup_{\lambda\in\mathcal{S}^+} epi(\lambda f_i)^*)$ and $\mathcal{B} := w^* - clconeco\{(\gamma_i, a_i) : i \in \mathcal{I};\ (0, 1)\}$. We are going to show that $\mathcal{A} = \mathcal{B}$. It is clear that $\mathcal{B} \subset \mathcal{A}$. To see the converse inclusion, let $(\gamma, a) \in \mathcal{A} = w^* - clco(\bigcup_{i\in\mathcal{I}} \bigcup_{\lambda\in\mathcal{S}^+} epi(\lambda f_i)^*)$. This means that $(\gamma, a) \in epi(\lambda f_i)^*$ for some $\lambda \in \mathcal{S}^+$ and some $i \in \mathcal{I}$. Since $(\lambda f_i)^*(\gamma) \le a$, then $\sup_{x\in X}[\gamma(x) - \lambda f_i(x)] \le a$. Thus, we conclude that

$$\sup_{x\in X}[(\gamma(x) - \lambda\gamma_i(x) + \lambda a_i)] \le a. \tag{3.1}$$

Moreover, we have $sup_{x\in X}(\gamma - \lambda\gamma_i)(x) \le 0$. This implies that $\gamma = \lambda\gamma_i$. Now, replace γ by $\lambda\gamma_i$ in (3.1), we obtain $a - \lambda a_i \ge 0$. Hence, $(\gamma, a) = (\lambda\gamma_i, \lambda a_i) + (0, a - \lambda a_i)$. Then, $(\gamma, a) \in \mathcal{B}$, which implies that $\mathcal{A} \subset \mathcal{B}$, and the proof is complete. □

Now, we define the reverse-convex set G by

$$G = \{x \in X : h_j(x) \in \mathcal{S},\quad j = 1, 2, \ldots, m\}.$$

It is clear that $G = X \setminus \bigcup_{j=1}^m G_j$, where $G_j = \{x \in X : h_j(x) \in -int\mathcal{S}\}\quad (j = 1, 2 \ldots, m)$ and $\mathcal{S} \cap (-\mathcal{S}) = 0$.

Let $F = \{x \in X : f_i(x) \in -\mathcal{S},\ \forall\ i \in \mathcal{I}\}$. Clearly that $F \subset G$ if and only if $F \bigcap(\bigcup_{j=1}^m G_j) = \emptyset$, that is, $\{x \in X : f_i(x) \in -\mathcal{S},\ \forall\, i \in \mathcal{I},\ h_j(x) \in -int\mathcal{S},$ for some $j = 1, 2, \ldots, m\} = \emptyset$. Therefore, the characterization of the set containments will be changed into existence theorem, where G is a reverse-convex set and $\mathcal{S} \cap (-\mathcal{S}) = 0$. Moreover, in such case we shall characterize the existence theorem in Sect. 5.

In the following, we explain some properties of the weak dual cone. The proof of the following lemma is easy, and therefore we omit it.

Lemma 3.2 *Let F be a closed convex set in X and $K^{\le} = \{(\gamma, b) \in X^* \times \mathbb{R} : \gamma(x) \le b,\ \forall\ x \in F\}$. Then*

$$F = \{x \in X : \gamma(x) \le b,\ \forall\ (\gamma, b) \in K^{\le}\}.$$

The next result is a consequence of the non-homogeneous Farkas Lemma (Lemma 3.1).

Proposition 3.2 *Let $F = \{x \in X : \gamma_t \le b_t,\ t \in \mathcal{J}\}$, where $\gamma_t : X \to \mathbb{R}$ is a continuous linear mapping and $b_t \in \mathbb{R}$. Then*

$$K^{\le} = w^* - cl(coneco\{(\gamma_t, b_t) : t \in \mathcal{J};\ (0, 1)\}).$$

It is clear that $cone\{(0,1)\} \subseteq K^{\leq}$, and Lemma 3.2 shows that the equality holds if and only if $F = X$. Let $G \neq \emptyset \neq F$ be closed convex sets with weak dual cone $K^{\leq}$ and $M^{\leq}$, respectively. According to the Lemma 3.2 and Proposition 3.2, we have $F \subset G$ if and only if $M^{\leq} \subset K^{\leq}$.

Proposition 3.3 *Let $F = \bigcap_{i \in \mathcal{I}} F_i \neq \emptyset$, where F_i is a closed convex set with weak dual cone $K_i^{\leq}$ $(i \in \mathcal{I})$. Then $K^{\leq} = w^* - clco(\bigcup_{i \in \mathcal{I}} K_i^{\leq})$.*

Proof Since $F_i = \{x \in X : \gamma(x) \leq b,\ \forall\ (\gamma, b) \in K_i^{\leq}\}$ $(i \in \mathcal{I})$, it follows that $F = \{x \in X : \gamma(x) \leq b,\ \forall\ (\gamma, b) \in K_i^{\leq}\}$. Then, in view of Proposition 3.2 and that $(0,1) \in K_i^{\leq}$ $(i \in \mathcal{I})$, we conclude that

$$K^{\leq} = w^* - cl(coneco[\bigcup_{i \in \mathcal{I}} K_i^{\leq} \cup \{(0,1)\}]) = w^* - clco \bigcup_{i \in \mathcal{I}} K_i^{\leq},$$

which completes the proof. □

Proposition 3.4 *Let $F = \{x \in X : f_t(x) \in -\mathcal{S},\ \forall\ t \in \mathcal{J}\} \neq \emptyset$, where $f_t : X \longrightarrow Z$ is a continuous $\mathcal{S}$-convex function. Then the weak dual cone of F is given by*

$$K^{\leq} = w^* - clco \bigcup_{t \in \mathcal{J}} \bigcup_{\lambda \in \mathcal{S}^+} epi(\lambda f_t)^*.$$

Proof This is an immediate consequence of Proposition 3.1 (note that $K^{\leq} = epi\sigma_F$). □

4 The containments of evenly convex sets in Banach spaces

The definition of the *strict dual cone* of a non-empty evenly convex set in $\mathbb{R}^n$ has been given in [5]. Let $F \subset X$ be a non-empty evenly convex set, the strict weak dual cone of F is denoted by:

$$K^{<} = \{(\gamma, b) \in X^* \times \mathbb{R} : \gamma(x) < b,\ \forall\ x \in F\}.$$

In the following, we explain some properties of the strict dual cones and evenly convex sets.

Lemma 4.1 *Let $\mathcal{A}$ be a subset of X, and let $y \in X$. Then $y \notin eco\mathcal{A}$ if and only if there exists $0 \neq \gamma \in X^*$ such that $\gamma(x - y) < 0$ for all $x \in \mathcal{A}$.*

Proof Let $y \notin eco\mathcal{A}$. By definition, there exists an open half space W such tat $\mathcal{A} \subset W$ and $y \notin W$. Now, by separation theorem there exists $0 \neq \gamma \in X^*$ such that $\gamma(x) < \gamma(y)$ for all $x \in W$. Hence $\gamma(x - y) < 0$ for all $x \in \mathcal{A}$.

Conversely, put $W = \{x \in X : \gamma(x) < \gamma(y)\}$. It is clear that $\mathcal{A} \subset W$ and $y \notin W$, and so $y \notin eco\mathcal{A}$. □

Lemma 4.2 *Let F be non-empty evenly convex set with associated strict dual cone $K^{<}$. Then*

$$F = \{x \in X : \gamma(x) < b, \quad \forall\ (\gamma, b) \in K^{<}\}.$$

Proof It is obvious that $F \subset \{x \in X : \gamma(x) < b, \quad \forall\ (\gamma, b) \in K^{<}\}$. To show the converse inclusion, let $x_0 \in \{x \in X : \gamma(x) < b, \quad \forall\ (\gamma, b) \in K^{<}\}$, and $x_0 \notin F$. Then, by Lemma 4.1, there exists $0 \neq \gamma \in X^*$ such that $\gamma(x) < \gamma(x_0)$ for all $x \in F$, and hence $\sup_{x \in F} \gamma(x) \leq \gamma(x_0)$. Put $b = \frac{1}{2}(\sup_{x \in F} \gamma(x) + \gamma(x_0))$. It is easy to see that $(\gamma, b) \in K^{<}$, which implies that $\gamma(x_0) < \sup_{x \in F} \gamma(x)$. This is a contradiction. □

In the rest of this section, we assume that F is a non-empty evenly convex set with associated strict dual cone $K^<$. Obviously, $\{0\} \times \mathbb{R}_{++} \subseteq K^<$ and the equality holds if and only if $F = X$. Since $0_{X^* \times \mathbb{R}} \notin K^<$, then $K^<$ cannot be weak *-closed. In particular, if F is closed, then we have $K^<$ contained strictly in $K^\leq$.

We denote the weak dual cone of $cl F$ by $\overline{K}^\leq$. It is clear that $\overline{K}^\leq$ is a weak*-closed convex cone and $\overline{K}^\leq = K^\leq$, if F is closed. The next result has been obtained in [5] in finite dimensional case. However, the same proof holds for the case under consideration.

Proposition 4.1 $w^* - cl K^< = \overline{K}^\leq$.

Example 4.1 If $F = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq y\}$, then $\overline{K}^\leq = K^\leq = \{(a, b) \in \mathbb{R}^2 : a + b \leq 0,\ b \leq 0\} \times \mathbb{R}_+$, and $K^< = \{(a, b) \in \mathbb{R}^2 : a + b \leq 0,\ b \leq 0\} \times \mathbb{R}_{++}$. Moreover, $w^* - cl K^< = \overline{K}^\leq$.

Proposition 4.2 *The cone $K^< \cup \{0_{X^* \times \mathbb{R}}\}$ is weak*-closed if and only if the set F is open. In such case, $K^< = \overline{K}^\leq \backslash \{0_{X^* \times \mathbb{R}}\}$.*

Proof Assume if possible that F is not open. Let $\overline{x} \in bd F \cap F$. Since F is convex and $\overline{x} \notin int F$, then by separation theorem there exists $0 \neq \gamma \in X^*$ such that $\gamma(x) < \gamma(\overline{x})$ for all $x \in int F$, which implies that $\sup_{x \in F} \gamma(x) \leq \gamma(\overline{x})$.

Put $b = \gamma(\overline{x})$. Let $n \in \mathbb{N}$ be arbitrary. Then, we have $\gamma(x) < b + \frac{1}{n}$ for all $x \in F$. Therefore, $(\gamma,\ b + \frac{1}{n}) \in K^< \subset K^< \cup \{0_{X^* \times \mathbb{R}}\}$ for all $n \in \mathbb{N}$, which implies that $(\gamma, b) \in w^* - cl(K^< \cup \{0_{X^* \times \mathbb{R}}\})$. On the other hand, $(\gamma, b) \notin K^<$ and $\gamma \neq 0$. Hence, $(\gamma, b) \notin K^< \cup \{0_{X^* \times \mathbb{R}}\}$. This implies that the cone $\overline{K}^\leq \cup \{0_{X^* \times \mathbb{R}}\}$ is not weak*-closed, which is a contradiction.

Conversely, suppose that $K^< \cup \{0_{X^* \times \mathbb{R}}\}$ is not weak*-closed. Choose $\{(\gamma_\alpha, b_\alpha)\}_{\alpha \in I} \subset K^< \cup \{0_{X^* \times \mathbb{R}}\}$ such that $(\gamma_\alpha, b_\alpha) \longrightarrow (\gamma, b)$ in the weak* topology, and $(\gamma, b) \notin K^< \cup \{0_{X^* \times \mathbb{R}}\}$. Let $\overline{x} \in F$ be such that $\gamma(\overline{x}) \geq b$. Since

$$\gamma_\alpha(\overline{x}) < b_\alpha \quad \forall\, \alpha \in I,$$

it follows that $\gamma(\overline{x}) = b$. By a similar argument, we have $\gamma(x) \leq b$ for all $x \in F$. Thus, $F \subset \{x \in X : \gamma(x) \leq b\}$. Since $bd\{x \in X : \gamma(x) \leq b\} = \{x \in X : \gamma(x) = b\}$, we get $\overline{x} \in bd F$, which implies that F cannot be open. This is a contradiction. Therefore, in view of Proposition 4.1, we conclude that $K^< = \overline{K}^\leq \backslash \{0_{X^* \times \mathbb{R}}\}$. □

Corollary 4.1 *If $|\mathcal{I}| < \infty$. Then, $\gamma(x) < b$ is a consequence of the consistent system $\{\gamma_t(x) < b_t,\ t \in \mathcal{I}\}$ if and only if*

$$(\gamma, b) \in [coneco\{(\gamma_t, b_t) : t \in \mathcal{I}; (0, 1)\}] \backslash \{0_{X^* \times \mathbb{R}}\}.$$

Proof Since $|\mathcal{I}| < \infty$, we have $F = \{x \in X : \gamma_t(x) < b_t,\ t \in \mathcal{I}\}$ is an open subset of X. In view of Proposition 3.3 we conclude that $\overline{K}^\leq = coneco\{(\gamma_t, b_t) : t \in \mathcal{I};\ (0, 1)\}$. Hence the conclusion follows from Proposition 4.2. □

Proposition 4.3 *If $K^<$ is relatively w^*-open, then F is closed.*

Proof Assume that F is not closed. Let $\{x_n\} \subset F$ be such that $\lim_n x_n = \overline{x} \notin F$. Since F is an evenly convex set, by Lemma 4.1, there exists $0 \neq \gamma \in X^*$ and $b \in \mathbb{R}$ such that $\gamma(x) < b$ for all $x \in F$ and $\gamma(\overline{x}) = b$. Obviously, $(\gamma, b) \in K^<$.

Let $\epsilon > 0$ be given. Since $\lim_n \gamma(x_n) = \gamma(\overline{x}) = b > b - \epsilon$, it follows that there exists $m \in \mathbb{N}$ such that $\gamma(x_m) > b - \epsilon$ with $x_m \in F$. Moreover, let $\{x_1, x_2, \ldots, x_n\} \subset X$ be arbitrary. Consider the set

$$\mathcal{V}_\epsilon^{x_1,x_2,\ldots,x_n}(\gamma) = \bigcap_{i=1}^{n} \{\Lambda \in X^* : |\Lambda(x_i) - \gamma(x_i)| < \epsilon\},$$

and

$$\mathcal{B}_\epsilon(b) = \{a \in \mathbb{R} : |a - b| < \epsilon\}.$$

On the other hand, $\mathcal{V}_\epsilon^{x_1,x_2,\ldots,x_n}(\gamma) \times \mathcal{B}_\epsilon(b)$ is an arbitrary w^*-open neighborhood of (γ, b). Now, since $(\gamma, \ b - \epsilon) \notin K^<$, and $(\gamma, \ b - \varepsilon) = (\gamma, b) - \varepsilon(0, 1) \in aff K^<$, we get $(\gamma, b - \epsilon) \in aff K^<$, and $[\mathcal{V}_\epsilon^{x_1,x_2,\ldots,x_n}(\gamma) \times \mathcal{B}_\epsilon(b)] \cap aff K^< \not\subset K^<$, where $aff K^<$ is the affine hull of $K^<$. Hence (γ, b) does not belong to the relative w^*-interior of $K^<$. This is a contradiction. □

Example 4.1 shows that the converse of Proposition 4.3 is not true. Next, we show that the compactness of F guarantees the w^*-openness of $K^<$.

Proposition 4.4 *If F is compact, then $K^<$ is w^*-open. In such case, $K^< = w^* - int\overline{K}^{\leq}$.*

Proof Since F is compact, σ_F is continuous on X. Let $(\gamma, b) \in K^<$. But, we have $\sigma_F(\gamma) = \max_{x \in F} \gamma(x) < b$, it follows that there exists $\epsilon > 0$ such that $\sigma_F(\gamma) < b - \epsilon$. By continuity of σ_F, we conclude that there exists a w^*-open set $\mathcal{V}_\delta^{x_1,x_2,\ldots,x_n}(\gamma) \times \mathcal{B}_\delta(b)$ of $X^* \times \mathbb{R}$, for some $0 < \delta < \frac{\epsilon}{2}$ and some finite set $\{x_1, x_2, \ldots, x_n\} \subset X$ such that if $(\Lambda, d) \in \mathcal{V}_\delta^{x_1,x_2,\ldots,x_n}(\gamma) \times \mathcal{B}_\delta(b)$, then we have

$$|\sigma_F(\Lambda) - \sigma_F(\gamma)| < \frac{\epsilon}{2}, \text{ whenever } |b - d| < \delta < \frac{\varepsilon}{2}.$$

Now, we are going to show that

$$\mathcal{V}_\delta^{x_1,x_2,\ldots,x_n}(\gamma) \times \mathcal{B}_\delta(b) \subset K^<.$$

To do this, let $(\Lambda, d) \in \mathcal{V}_\delta^{x_1,x_2,\ldots,x_n}(\gamma) \times \mathcal{B}_\delta(b)$. Thus, $\sigma_F(\Lambda) < \frac{\epsilon}{2} + \sigma_F(\gamma) < b - \epsilon + \frac{\epsilon}{2} < d$, so that $(\Lambda, d) \in K^<$. Therefore, $K^<$ is w^*-open, and hence $K^< = w^* - int\overline{K}^{\leq}$. □

Proposition 4.4 yields another version of Farkas Lemma for linear strict inequalities.

Corollary 4.2 *If the solution set of the system $\{\gamma_t(x) < b_t, \ \ \forall \ t \in \mathcal{I}\}$ is compact, then $\gamma(x) < b$ is a consequence of this system if and only if*

$$(\gamma, b) \in w^* - int[\, coneco\{(\gamma_t, b_t) : \ t \in \mathcal{I}; \ (0, 1)\}].$$

Proof This is an immediate consequence of Proposition 4.4. □

One of the important characterizations of solvability theorem is general Farkas Lemma for systems of strict inequalities. Proposition 5.5 in [5] gives a condition for the systems of strict inequalities, but its proof is false. On the other hand, in the first part of the proof of Proposition 5.5 in [5], we see that "in both cases, there exists a hyperplane containing (a, b) and 0_{n+1}, which does not contain points of X." This sentence is the main key of the proof. The following example shows that this statement is not true.

Example 4.2 Consider the system $X = \{(x, y) \in \mathbb{R}^2 : \ (1, -1)(x, y) < 0, \ (1, 1)(x, y) < 0\}$. Let $a_1 = (1, -1), \ a_2 = (1, 1), \mathcal{I} = \{1, 2\}$ and $(a, b) = (-1, 1, 0)$. Now, for every hyperplane H containing $(-1, 1, 0)$ and 0_3, we have H contains $(1, -1, 0)$, but $(1, -1, 0) \in X$.

The next result is the general Farkas Lemma for systems of strict inequalities in infinite dimensional case.

Theorem 4.1 *Let $\mathcal{I}$ be an arbitrary index set and $F = \{x \in X \ : \ \gamma_t(x) < b_t, \ \ \forall \ t \in \mathcal{I}\} \neq \emptyset$, where $\gamma_t : X \longrightarrow \mathbb{R}$ is a continuous linear map. Assume that $u : X \longrightarrow \mathbb{R}$ is a continuous linear map. Then the following assertions are equivalent:*

(i) $\{x \in X : \gamma_t(x) < b_t, \ \forall \ t \in \mathcal{I}\} \subset \{x \in X : u(x) < \beta\}$.
(ii) $(u, \beta) \in eco[\mathbb{R}_{++}\{(\gamma_t, b_t) : \ \forall \ t \in \mathcal{I}; \ \ (0, 1)\}]$.

Proof Let $A = \mathbb{R}_{++}\{(\gamma_t, b_t) : \ \forall \ t \in \mathcal{I}; \ \ (0, 1)\}$.
(i) $\Rightarrow$ (ii): Suppose that $(u, \beta) \notin ecoA$. Then, by Lemma 4.1, there exists $(y, d) \in X \times \mathbb{R}$ such that

$$(y, d)[(\Lambda, b) - (u, \beta)] < 0 \quad \forall \ (\Lambda, b) \in A. \tag{4.1}$$

Thus, we have

$$\lambda(\Lambda(y) + db) < u(y) + d\beta \quad \forall \ (\Lambda, b) \in A \quad \text{and} \quad \forall \ \lambda > 0. \tag{4.2}$$

This implies that

(1) $u(y) + d\beta \geq 0$, as $\lambda \to 0$.
(2) $\Lambda(y) + db \leq 0$ for all $(\Lambda, b) \in A$, as $\lambda \to \infty$.

Also, by (4.2) for $\lambda = 1$, we have either

$$u(y) + d\beta \geq 0, \ \ \Lambda(y) + bd < 0 \quad \forall \ (\Lambda, b) \in A.$$

In this case, put $\overline{x}_0 := y$ and $c := d$, or

$$u(y) + d\beta > 0, \ \ \Lambda(y) + db \leq 0 \quad \forall \ (\Lambda, b) \in A.$$

Now, assume that $u(y) + d\beta > 0$ and $\Lambda(y) + db \leq 0$ for all $(\Lambda, b) \in A$. By hypothesis, we have $F = \{x \in X : \ \gamma_t(x) < b_t, \ \ \forall \ t \in \mathcal{I}\} \neq \emptyset$, so there exists $l \in X$ such that $\gamma_t(l) - b_t < 0$ for all $t \in \mathcal{I}$. This implies that

$$\Lambda(l) - b < 0, \quad \forall \ (\Lambda, b) \in A.$$

Moreover, since $u(y) + d\beta > 0$ and (i) implies that $(l, -1)(u, \beta) < 0$, then there exists $n \in \mathbb{N}$ such that $(y, d)(u, \beta) + \frac{1}{n}(l, -1)(u, \beta) \geq 0$. Also,

$$(y, d)(\Lambda, b) + \frac{1}{n}(l, -1)(\Lambda, b) < 0, \quad \forall \ (\Lambda, b) \in A. \tag{4.3}$$

This implies that

$$\left(\frac{1}{n}l + y, d - \frac{1}{n}\right)(\Lambda, b) < 0, \ \left(\frac{1}{n}l + y, d - \frac{1}{n}\right)(u, \beta) \geq 0 \quad \forall \ (\Lambda, b) \in A. \tag{4.4}$$

In this case, put $\overline{x}_0 := \frac{1}{n}l + y$ and $c := d - \frac{1}{n}$. Hence, in both the above cases, we have

$$(\overline{x}_0, c)(\Lambda, b) < 0 \ \forall \ (\Lambda, b) \in A, \quad \text{and} \quad (\overline{x}_0, c)(u, \beta) \geq 0. \tag{4.5}$$

Since $(0, 1) \in A$, it follows from (4.5) that $c < 0$. Let $\overline{x} := |c|^{-1}\overline{x}_0$. Multiplying by $|c|^{-1}$ each expression of (4.5), we obtain $\gamma_t(\overline{x}) < b_t$ for all $t \in \mathcal{I}$, and $u(\overline{x}) \geq \beta$, which is a contradiction.

(ii) $\Rightarrow$ (i): Assume if possible that $\gamma_t(\overline{x}) < b_t$ for all $t \in \mathcal{I}$ and $u(\overline{x}) \geq \beta$ for some $\overline{x} \in X$. This implies that $(\overline{x}, -1)(\Lambda, b) < 0$ for all $(\Lambda, b) \in A$ and $(\overline{x}, -1)(u, \beta) \geq 0$. Therefore, we have

$$(\overline{x}, -1)[(\Lambda, b) - (u, \beta)] < 0 \quad \forall\, (\Lambda, b) \in A. \tag{4.6}$$

This, together with Lemma 4.1 imply that $(u, \beta) \notin ecoA$, which completes the proof. □

Corollary 4.3 *Let $\mathcal{I}$ be an arbitrary index set and $F = \{x \in X : \gamma_t(x) < b_t, \forall\, t \in \mathcal{I}\} \neq \emptyset$, where $\gamma_t : X \longrightarrow \mathbb{R}$ is a continuous linear mapping. Then*

$$K^{<} = eco[\mathbb{R}_{++}\{(\gamma_t, b_t) : \ \forall\, t \in \mathcal{I};\ (0, 1)\}].$$

The proof of the following proposition is obvious, and therefore we omit it.

Proposition 4.5 *Let $F = \bigcap_{i\in\mathcal{I}} F_i \neq \emptyset$, where F_i is an evenly convex set with strict dual cone $K_i^{<}$ $(i \in \mathcal{I})$. Then, $K^{<} = eco[\bigcup_{i\in\mathcal{I}} K_i^{<}]$.*

The next result is the general non-linear Farkas Lemma for systems of strict inequalities in infinite dimensional case.

Corollary 4.4 *Let $F = \{x \in X : f_t(x) \in -int\mathcal{S}, \quad \forall\, t \in J\} \neq \emptyset$, where $f_t : X \longrightarrow Z$ is a continuous $\mathcal{S}$-convex function for each $t \in J$. Assume that $u : X \longrightarrow \mathbb{R}$ is a continuous linear map. Then the following assertions are equivalent:*

(i) $\{x \in X : f_t(x) \in -int\mathcal{S}, \ \forall\, t \in J\} \subset \{x \in X : u(x) < b\}$.
(ii) $(u, b) \in eco[(\bigcup_{t\in J} \bigcup_{\lambda\in\mathcal{S}^+} epi(\lambda f_t)^*)\backslash\{0_{X^*\times\mathbb{R}}\}]$.

Proof Suppose $F_t = \{x \in X : f_t(x) \in -int\mathcal{S}\}$ for each $t \in J$. Then, $F = \cap_{t\in J} F_t$. Therefore, by Proposition 3.2, the weak dual cone of clF_t $(clF_t := \{x \in X : f_t(x) \in -\mathcal{S}\})$ is $\overline{K_t}^{\leq} = epi\sigma_{clF_t} = w^* - cl(\bigcup_{\lambda\in\mathcal{S}^+} epi(\lambda f_t)^*)$. Since $F \neq \emptyset$, then $F_t \neq \emptyset$ for each $t \in J$. This implies that the Slater condition holds, that is, there exists $x_t \in X$ such that $f_t(x_t) \in -int\mathcal{S}$, thus the set $\bigcup_{\lambda\in\mathcal{S}^+} epi(\lambda f_t)^*$ is w^*-closed (see [9]). Therefore, we have

$$\overline{K_t}^{\leq} = \bigcup_{\lambda\in\mathcal{S}^+} epi(\lambda f_t)^*.$$

Then, according to Proposition 4.2, we obtain

$$K_t^{<} = (\bigcup_{\lambda\in\mathcal{S}^+} epi(\lambda f_t)^*)\backslash\{0_{X^*\times\mathbb{R}}\},$$

and hence, by Proposition 4.5, we get

$$K_t^{<} = eco[(\bigcup_{t\in J} \bigcup_{\lambda\in\mathcal{S}^+} epi(\lambda f_t)^*)\backslash\{0_{X^*\times\mathbb{R}}\}],$$

which completes the proof. □

5 General existence theorem with applications

It is well-known that every positively homogenous convex (sublinear function) lower semi-continuous function f is equal to support function C, where $C = \{\gamma \in X^* : \gamma(x) \leq f(x),\ \forall\ x \in X\} = \partial f(0)$. In this section, we show that $\partial f(0)$ is $weak^*$-compact. On the other hand, if f is a positively homogenous lower semi-continuous convex function, then $f(x) = \max_{\gamma \in \partial f(0)} \gamma(x)$.

Let $\mathbb{R}^X$ be the set of all functions defined on X with values in $\mathbb{R}$. Thus, $\mathbb{R}^X$ can be regarded as a topological product space.

Lemma 5.1 ([1], Corollary 2.2). *A set $M \subset X^*$ is $weak^*$-compact if and only if it is closed in $\mathbb{R}^X$, and for every $x \in X$, there exists $K_x > 0$ such that $|\gamma(x)| \leq K_x$ for all $\gamma \in M$.*

The following theorem and its corollary are well-known (see for example [13]). For an easy reference we gather another proof.

Theorem 5.1 *Let $f : X \longrightarrow \mathbb{R}$ be a positively homogenous lower semi-continuous convex function. Then, $\partial f(0)$ is $weak^*$-compact.*

Proof Let $x \in X$ be arbitrary. By definition of $\partial f(0)$, we have

$$\gamma(x) \leq f(x),\ \ \gamma(-x) \leq f(-x) \quad \forall\, \gamma \in \partial f(0).$$

Put, $K_x = \max\{f(x), f(-x)\}$. Thus, we have

$$\gamma(x) \leq K_x, \quad \text{and} \quad \gamma(-x) \leq K_x \quad \forall\, \gamma \in \partial f(0).$$

This implies that $|\gamma(x)| \leq K_x$ for all $\gamma \in \partial f(0)$. Now, by Lemma 5.1, we need only prove that $\partial f(0)$ is closed in $\mathbb{R}^X$. To do this, let $\theta_0 : X \longrightarrow \mathbb{R}$ be a limit point of $\partial f(0)$ in the product topology of $\mathbb{R}^X$. First, we show that θ_0 is linear. Let $x,\ \ y \in X$, $a \in \mathbb{R}$ and $\epsilon > 0$ be arbitrary. Consider the $\mathcal{V}_F(\theta_0)$ as in (2.1) with $F = \{x, y, ax + y\}$. By hypothesis, there exists $\theta \in \partial f(0) \cap \mathcal{V}_F(\theta_0)$ such that

$$|\theta(x) - \theta_0(x)| < \epsilon, \quad |\theta(y) - \theta_0(y)| < \epsilon, \quad \text{and} \quad |\theta(ax + y) - \theta_0(ax + y)| < \epsilon.$$

From the above relations, we obtain $|\theta_0(ax + y) - a\ \theta_0(x) - \theta_0(y)| < (2 + |a|)\epsilon$, and hence θ_0 is linear.

Also, we have

$$|\theta_0(x)| \leq |\theta(x)| + |\theta(x) - \theta_0(x)| \leq \|\theta\|\ \|x\| + \epsilon. \tag{5.1}$$

Moreover, since f is lower semi-continuous, we get $f(x) < \alpha$ for some $\alpha \in \mathbb{R}$ and for all $x \in \mathcal{B}$, where $\mathcal{B} = \{x \in X\ :\ \|x\| \leq 1\}$. Therefore, since $\theta \in \partial f(0)$, we conclude that

$$\theta(x) < \alpha \quad \forall\, x \in \mathcal{B}.$$

This implies that

$$\|\theta\| < \alpha. \tag{5.2}$$

Now, according to (5.1) and (5.2), we obtain

$$|\theta_0(x)| \leq \|\theta\|\ \|x\| + \epsilon < \alpha\|x\| + \epsilon \quad \forall\, x \in \mathcal{B}.$$

Hence, $\|\theta_0\| \leq \alpha$, and hence θ_0 is continuous. Finally, we must show that $\theta_0(x) \leq f(x)$ for all $x \in X$. Let $x \in X$ and $\epsilon > 0$ be arbitrary. Since θ_0 is a limit point of $\partial f(0)$ in the product topology of $\mathbb{R}^X$, it follows that there exists $\theta_x \in \partial f(0)$ such that $|\theta_x(x) - \theta_0(x)| < \epsilon$. Thus, we have

$$\theta_0(x) = \theta_x(x) + \theta_0(x) - \theta_x(x) \leq f(x) + |\theta_0(x) - \theta_x(x)| < f(x) + \epsilon.$$

This implies that $\theta_0(x) \leq f(x)$, and the proof is complete. □

Corollary 5.1 *Let $f : X \longrightarrow \mathbb{R}$ be a lower semi-continuous sublinear function. Then $f(x) = \max_{\gamma \in \partial f(0)} \gamma(x)$.*

The rest of this section concentrates on existence theorem for convex systems by strict reverse-convex inequalities. The next result is an extension of Proposition 2.1 in infinite dimensional case.

Proposition 5.1 *Let $\mathcal{I}$, $\mathcal{J}$ be arbitrary index sets, $\mathcal{I} \neq \emptyset$. The system $F = \{a_t(x) < b_t,\ t \in \mathcal{I};\ a_t(x) \leq b_t,\ t \in \mathcal{J}\}$ is consistent if and only if*

$$0_{X^* \times \mathbb{R}} \notin eco[\{(a_t, b_t) : t \in \mathcal{I}\} + \mathbb{R}_+\{(a_t, b_t) : t \in \mathcal{J}\};\ (0_{X^* \times \mathbb{R}}, 1)].$$

Proof Put $A = [\{(a_t, b_t) : t \in \mathcal{I}\} + \mathbb{R}_+\{(a_t, b_t) : t \in \mathcal{J}\};\ (0_{X^* \times \mathbb{R}}, 1)]$. Suppose that F is consistent, so there exists $\overline{x} \in F$ such that $(\overline{x}, -1)[(a_{t_1}, b_{t_1}) + \lambda(a_{t_2}, b_{t_2})] < 0$ for all $\lambda \geq 0$, $t_1 \in \mathcal{I}$ and all $t_2 \in \mathcal{J}$. According to Lemma 4.1 , we have $0_{X^* \times \mathbb{R}} \notin ecoA$.

Conversely, assume that $0_{X^* \times \mathbb{R}} \notin ecoA$. Then, by Lemma 4.1, there exists $(\overline{x}_0, d) \in X \times \mathbb{R}$ such that

$$(\overline{x}_0, d)[(a_{t_1}, b_{t_1}) + \lambda(a_{t_2}, b_{t_2})] < 0 \qquad \forall\, \lambda \geq 0,\ t_1 \in \mathcal{I},\ t_2 \in \mathcal{J}.$$

Since $(0, 1) \in A$, we get $d < 0$. Put $\overline{x} := -\frac{\overline{x}_0}{d}$. Thus

$$(\overline{x}, -1)[(a_{t_1}, b_{t_1}) + \lambda(a_{t_2}, b_{t_2})] < 0 \qquad \forall\, \lambda \geq 0,\ t_1 \in \mathcal{I},\ t_2 \in \mathcal{J}. \tag{5.3}$$

Moreover, by letting $\lambda = 0$ in (5.3), we have $(\overline{x}, -1)(a_{t_1}, b_{t_1}) < 0$ for all $t_1 \in \mathcal{I}$. Also, it follows from (5.3) that $(\overline{x}, -1)(a_{t_2}, b_{t_2}) \leq 0$ for all $t_2 \in \mathcal{J}$. Hence, $\overline{x} \in F$, which completes the proof. □

Proposition 5.2 *Let $F = \{x \in X : f_t(x) < 0,\ t \in \mathcal{I};\ f_t(x) \leq 0,\ t \in \mathcal{J}\}$, where $\mathcal{I} \neq \emptyset$, $f_t(x) = g_t(x) - b_t$, $b_t \in \mathbb{R}$, with $g_t : X \longrightarrow \mathbb{R}$ is a continuous sublinear function for each $t \in \mathcal{I} \cup \mathcal{J}$. Then the system F is consistent if and only if*

$$0_{X^* \times \mathbb{R}} \notin eco\{(\bigcup_{t \in \mathcal{I}} \partial f_t(0) \times \{b_t\}) + \mathbb{R}_+(\bigcup_{t \in \mathcal{J}} \partial f_t(0) \times \{b_t\});\ (0_{X^*}, 1)\}.$$

Proof This is an immediate consequence of Corollary 5.1 and Proposition 5.1. □

Lemma 5.2 *Let X be a Banach space and $\mathcal{S}$ be a closed convex cone in X such that $int\mathcal{S} \neq \emptyset$. Then*

$$int\mathcal{S} = \{x \in X : \quad \lambda(x) > 0,\ \forall\, \lambda \in \mathcal{S}^+\backslash\{0\}\}.$$

Proof Since

$$\mathcal{S} = \{x : \lambda(x) \geq 0,\ \forall\, \lambda \in \mathcal{S}^+\},$$

it follows that int $\mathcal{S} \supset \{x \in X: \lambda(x) > 0,\ \forall\, \lambda \in \mathcal{S}^+\backslash\{0\}\ \}$. For the converse inclusion, let $x \in int\mathcal{S}$. Then there exists $r > 0$ such that the neighborhood $\mathcal{N}_r(x) \subset \mathcal{S}$. This implies that $\lambda(u) \geq 0$ for all $u \in \mathcal{N}_r(x)$, and all $\lambda \in \mathcal{S}^+\backslash\{0\}$. Assume that $x \notin \{x \in X : \lambda(x) > 0,\ \forall\, \lambda \in$

$\mathcal{S}^+\backslash\{0\}$ }. That is, there exists a $\lambda \in \mathcal{S}^+\backslash\{0\}$ such that $\lambda(x) = 0$. Let $0 \neq u \in \mathcal{N}_r(x)$. It is clear that $x + \frac{u}{\|u\|}\frac{r}{2}$ and $x - \frac{u}{\|u\|}\frac{r}{2} \in \mathcal{N}_r(x)$. Thus, $\lambda(x + \frac{u}{\|u\|}\frac{r}{2}) \geq 0$ and $\lambda(x - \frac{u}{\|u\|}\frac{r}{2}) \geq 0$. This, together with the fact $\lambda(x) = 0$ imply that $\lambda(u) = 0$. Thus, we have $\lambda(u) = 0$ for all $u \in \mathcal{N}_r(x)$. Hence, $\lambda = 0$ on X. This is a contradiction, and the proof is complete. □

The next result is the general existence theorem for the system of strict reverse-convex inequality with the collection of sublinear functions and reverse-convex inequality with the collection of convex functions.

Theorem 5.2 *Let $F = \{x \in X : f_t(x) \in -int\mathcal{S},\ t \in \mathcal{I};\quad f_t(x) \in -\mathcal{S},\ t \in \mathcal{J}\}$, where $\mathcal{S}$ is a non-empty closed convex cone in Z, $f_t : X \longrightarrow Z$ is a continuous $\mathcal{S}$-convex function for each $t \in \mathcal{I} \cup \mathcal{J}$, and f_t is a positively homogenous function for each $t \in \mathcal{I}$ with $\mathcal{I} \neq \emptyset$. Then, $F \neq \emptyset$ if and only if*

$$0_{X^*\times\mathbb{R}} \notin eco\{\bigcup_{t\in\mathcal{I}} \bigcup_{\lambda\in\mathcal{S}^+\backslash\{0\}} epi(\lambda f_t)^* + \mathbb{R}_+(\bigcup_{t\in\mathcal{J}} \bigcup_{\lambda\in\mathcal{S}^+} epi(\lambda f_t)^*);\ (0_{X^*}, 1)\}.$$

Proof Let $A = \{\bigcup_{t\in\mathcal{I}} \bigcup_{\lambda\in\mathcal{S}^+\backslash\{0\}} epi(\lambda f_t)^* + \mathbb{R}_+(\bigcup_{t\in\mathcal{J}} \bigcup_{\lambda\in\mathcal{S}^+} epi(\lambda f_t)^*);\ (0_{X^*}, 1)\}$. Suppose that $0_{X^*\times\mathbb{R}} \notin ecoA$. Then, by Lemma 4.1, there exists $(\overline{x}_0, d) \in X \times \mathbb{R}$ such that $(\overline{x}_0, d)[(\gamma_t, b_t) + \mu(\Lambda_s, d_s)] < 0$, for all $\mu \geq 0,\ t \in \mathcal{I},\ s \in \mathcal{J},\ (\gamma_t, b_t) \in epi(\lambda f_t)^*$ and all $(\Lambda_s, d_s) \in epi(\lambda f_s)^*$. By a similar argument as in the proof of Proposition 5.1, we obtain

$$(\overline{x}_0, -1)[(\gamma_t, b_t) + \mu(\Lambda_s, d_s)] < 0, \tag{5.4}$$

for all $\mu \geq 0,\ t \in \mathcal{I},\ s \in \mathcal{J},\ (\gamma_t, b_t) \in epi(\lambda f_t)^*$ and all $(\Lambda_s, d_s) \in epi(\lambda f_s)^*$. Let $\lambda \in \mathcal{S}^+\backslash\{0\},\ t \in \mathcal{I}$ and $(\gamma_t, b_t) \in epi(\lambda f_t)^*$ be arbitrary. Put, $b_t = (\lambda f_t)^*(\gamma_t)$ and $\mu = 0$ in (5.4). Then we have

$$\gamma_t(\overline{x}_0) < (\lambda f_t)^*(\gamma_t). \tag{5.5}$$

Since λf_t is continuous, it follows that

$$(\lambda f_t)(\overline{x}_0) = (\lambda f_t)^{**}(\overline{x}_0) = \sup_{\gamma_t\in dom(\lambda f_t)^*} [\gamma_t(\overline{x}_0) - (\lambda f_t)^*(\gamma_t)]. \tag{5.6}$$

It is easy to see that $dom(\lambda f_t)^* = \partial(\lambda f_t)(0)$. So, by Theorem 5.1, $dom(\lambda f_t)^*$ is weak*-compact. Thus, we have

$$\sup_{\gamma_t\in dom(\lambda f_t)^*} [\gamma_t(\overline{x}_0) - (\lambda f_t)^*(\gamma_t)] = \max_{\gamma_t\in dom(\lambda f_t)^*} [\gamma_t(\overline{x}_0) - (\lambda f_t)^*(\gamma_t)].$$

According to this fact and (5.5)and (5.6), we get $(\lambda f_t)(\overline{x}_0) < 0$. Hence, by Lemma 5.2, we obtain $f_t(\overline{x}_0) \in -int\mathcal{S}$ for all $t \in \mathcal{I}$.

Now, let $\lambda \in \mathcal{S}^+,\ s \in \mathcal{J}$ and $(\Lambda_s, d_s) \in epi(\lambda f_s)^*$ be arbitrary. This implies that $\Lambda_s(\overline{x}_0) \leq d_s$. Put $d_s = (\lambda f_s)^*(\Lambda_s)$. Hence, $\Lambda_s(\overline{x}_0) - (\lambda f_s)^*(\Lambda_s) \leq 0$ for all $\Lambda_s \in dom(\lambda f_s)^*$. Thus

$$(\lambda f_s)(\overline{x}_0) = (\lambda f_s)^{**}(\overline{x}_0) = \sup_{\Lambda_s\in dom(\lambda f_s)^*} [\Lambda_s(\overline{x}_0) - (\lambda f_s)^*(\Lambda_s)] \leq 0.$$

This shows that $f_s(\overline{x}_0) \in -\mathcal{S}$ for all $s \in \mathcal{J}$, and hence $F \neq \emptyset$.

Conversely, assume that $F \neq \emptyset$. Then there exists $\overline{x} \in F$ such that in view of Lemma 5.2, we obtain

$$-\lambda f_t(\overline{x}) > 0,\ and\ -\mu f_s(\overline{x}) \geq 0,\ \forall\, t \in \mathcal{I},\ s \in \mathcal{J},\ \lambda \in \mathcal{S}^+\backslash\{0\},\ \mu \in \mathcal{S}^+. \tag{5.7}$$

Now, we are going to show that $(\overline{x}, -1)(\gamma, b) < 0$ for all $(\gamma, b) \in A$. To do this, let $\lambda \in \mathcal{S}^+\backslash\{0\}$, $\mu \in \mathcal{S}^+$, $t \in \mathcal{I}$, $s \in \mathcal{J}$, $(\gamma_t, b_t) \in epi(\lambda f_t)^*$ and $(\Lambda_s, d_s) \in epi(\mu f_s)^*$ be arbitrary. Then, according to (5.7), we have

$$\Lambda_s(\overline{x}) + \gamma_t(\overline{x}) < \Lambda_s(\overline{x}) + \gamma_t(\overline{x}) - \mu f_s(\overline{x}) - \lambda f_t(\overline{x}) \leq d_s + b_t.$$

This implies that

$$(\overline{x}, -1)(\Lambda_s + \gamma_t,\ d_s + b_t) < 0.$$

In view of Lemma 4.1, we obtain $0_{X^* \times \mathbb{R}} \notin ecoA$, and the proof is complete. □

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